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October 9, 1944

AMG - NYU No. 71

Dr. Jesse R. D. Dulong
California Institute of Technology
Physics Department
Pasadena 4, Calif.

Dear Dr. Dulong:

I am sending you a short note concerning the problem of the momentum and energy balance in the wave produced by a flying bullet.

In our discussion the following points were made. If the bullet flies with constant speed, the pressure it exerts against the air does work and hence energy is fed at a constant rate into the wave. Flight with constant speed could therefore be maintained only if energy is constantly fed into the bullet. If that is not done, the bullet will be decelerated and the amount of energy lost to the wave will decrease. In order to check the momentum and energy balance with experiment, it is probably a sufficient approximation to consider the bullet speed momentarily constant.

A part of the energy fed into the wave is transformed into heat owing to the viscosity acting in the shock fronts. This effect of viscosity is accounted for in the Rankine-Hugoniot relations; it can thus not be ignored. The action of viscosity between the two shock fronts,

however, can safely be neglected, I should think.

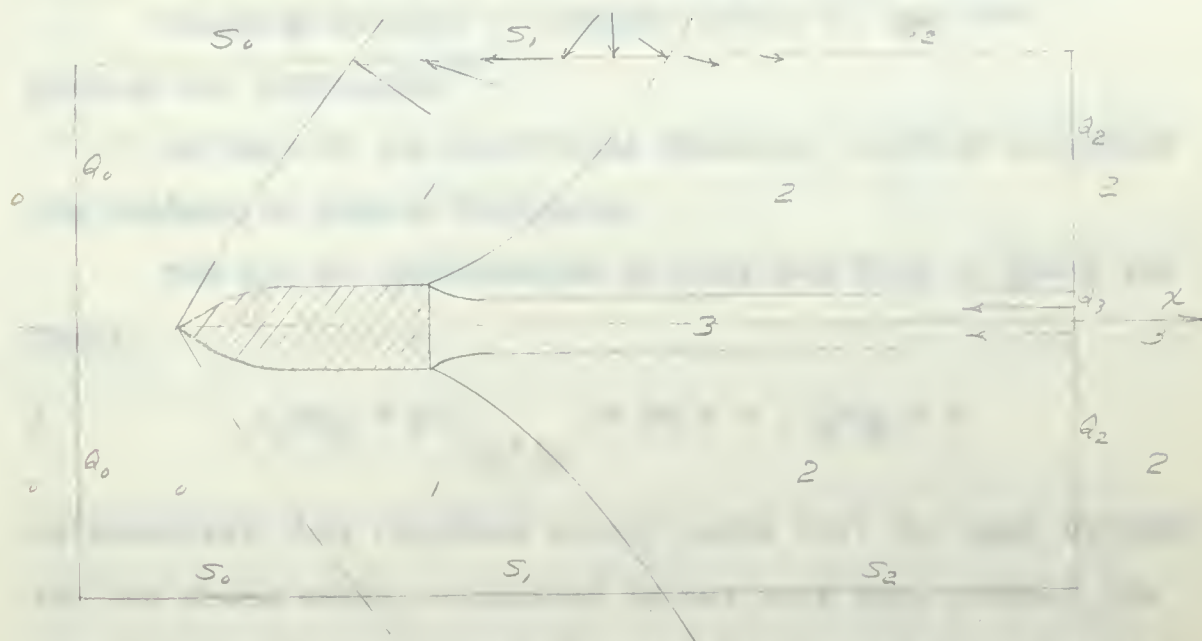
Let us consider a rotationally symmetric bullet which flies with constant speed. The resulting flow is then also rotationally symmetric. This flow is characterized by a "first" shock front, which begins at the tip of the bullet or in front of it; by a "second" shock front, which begins near the rim of the end surface of the bullet; and the wake, which begins at the end surface and whose cross-section decreases, eventually approaching a constant area.

The air is at rest in front of the first shock front; the same is assumed to be true sufficiently far behind the second shock front. Also, the pressure sufficiently far behind the second shock front is assumed to equal the pressure p_0 in front of the first shock front. The wake is not well delimited in reality and the flow in it is turbulent. Nevertheless, one may probably assume with good accuracy that the velocity in the wake is the same as that of the bullet.

We place the x-axis along the flight path, positive in the direction opposite to that of flight; the distance from the axis is denoted by r . Let U be the speed of the bullet, u and v the axial and the radial component of the air velocity positive in direction of increasing x and r respectively. The density of the air is ρ , the pressure

p , the internal energy p. u. mass e , and the enthalpy p. u. mass is $1 = e + p/\rho$. The force exerted by the bullet against the air in direction of flight is denoted by F .

By (0), (1), (2), (3) we characterize the regions in front of the first shock front, between the two shock fronts, behind the second shock front, and the wake region.



To formulate the laws of conservation of mass, momentum, and energy it is most convenient to consider a cylindrical "test box" as indicated in the diagram. The surface of the box, the "test surface", consists of the lateral sections S_0 , S_1 , S_2 , and the cross-sections Q_0 , Q_2 , and Q_3 , the subscript corresponding with the labels of the regions in which

these sections lie. The cross-section Q_2 is supposed to lie so far behind the second shock front that the velocity there can be assumed to be zero and the pressure to be equal to p_0 . By Q_0, Q_2, Q_3 we also denote the areas of the surfaces Q_0, Q_2, Q_3 . By $p_0, \rho_0, p_2 = p_0, \rho_2$, we denote pressure and density in region (o) and on surface Q_2 respectively.

Velocity vectors at various points of the test surface are indicated.*

In view of the rotational symmetry, surface integrals are reduced to simple integrals.

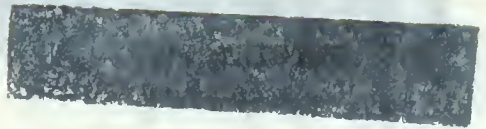
The law of conservation of mass can then be given the form:

$$I \quad -\rho_0 U Q_0 + 2\pi \int_{S_1+S_2} \rho v r dx + \rho_2 U Q_2 = 0$$

To interpret this relation we may argue that the mass between the two shock waves, contained in the test box, remains the same. Hence the decrease of mass in the part of the box in front of the first shock front, the first term** in relation I, equals the sum of the increase of mass in the part of the box behind the second shock front, the third term in I, and the

*It is clear that the vector behind the first shock front is perpendicular to it since this front moves into quiet air. In front of the second shock front the vector is again directed toward this front nearly perpendicular to it since the air behind it is nearly brought to rest. Consequently, the velocity vector turns through 180° along S_1 ; it is doubtful though whether it turns in the positive sense as indicated in the figure or in the negative sense.

** We denote by "term" an expression without the sign in front of it.



net mass flux p. u. time out of the box across, $S_1 + S_2$, the second term in I. (Incidentally, the second term is positive since $u_0 > u_2$ and $\rho_0 > \rho_2$ as explained a little later.)

The law of conservation of momentum can be given the form:

$$II \quad -2\pi \int_{S_1+S_2} \rho uv \, dx = F$$

The left hand side is the total momentum in flight direction flowing out of the section $S_1 + S_2$ per unit time.

The law of conservation of energy can be expressed by

$$-2\pi i_0 \rho_0 u_0 + 2\pi \int_{S_1+S_2} \left(1 + \frac{1}{2}u^2 + \frac{1}{2}v^2\right) \rho v \, dx + 2\pi i_2 \rho_2 u_2 = Fv.$$

The second term represents the sum of enthalpy and kinetic energy, which flow out of sections $S_1 + S_2$, the third and first term represent gain and loss of enthalpy p. u. time in the region immediately behind the second and in front of the first shock front respectively. Clearly, the net gain and outflow p. u. time of enthalpy and kinetic energy equals the power of the force exerted by the bullet.

For polytropic gases one has

$$i^p = \frac{\gamma}{\gamma-1} p,$$

hence $i^p = i_0 \rho_0$ on Q_2 ; the third term in III can therefore

be combined with the first one to give

$$\text{III}' \quad 2\pi \int_{S_1+S_2} \left(1 + \frac{1}{2}u^2 + \frac{1}{2}v^2\right) \rho v \, dS - 2\pi i_0 \rho_0 U_0 = FV$$

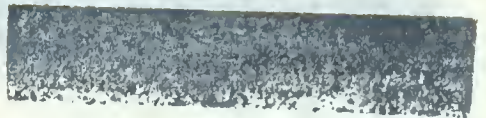
A similar simplification is not possible with relation I since the density ρ_2 behind the second shock front is less than ρ_0 , the density in front of the first shock front. This follows from the fact that for the air in region (2), having crossed two shock fronts, the entropy is greater than in region (0); hence p/ρ^γ is greater in (2) than in (0) and since $p_2 = p_0$ it follows that $\rho_2 < \rho_0$.

It may further be noted that the velocity (u,v) in region (2) is rather small and that the integrals extending over section S_2 may probably be neglected.

For a check of relations II and III' it is therefore essentially only necessary to know the quantities ρ, i, u , and v on the segment S_1 . The quantities ρ and i can be easily (approximately) determined from the experimentally found pressure distribution along S_1 . To this end one may make use of the formula

$$\frac{\rho_1}{\rho_0} = \frac{\frac{p_1}{p_0} + \mu^2}{\mu^2 \frac{p_1}{p_0} + 1}, \quad \mu^2 = \frac{\gamma - 1}{\gamma + 1}$$

where the subscript 0 refers to the region (0) in front of the first shock front, the subscript 1 to the back side of



this shock front (and not to the whole region (1) behind it).

We have set

$$\mu^2 = \frac{\gamma - 1}{\gamma + 1};$$

for air with $\gamma = 1.4$ we have $\mu^2 = \frac{1}{6}$. If $\frac{p_1}{p_0}$ differs little from one, we have

$$\frac{\rho_1 - \rho_0}{\rho_0} = \frac{1}{\gamma} \left(\frac{p_1 - p_0}{p_0} \right) = \frac{5}{7} \frac{p_1 - p_0}{p_0}.$$

The relative rise in entropy is given by

$$\frac{p_1 \rho_1^{-\gamma} - p_0 \rho_0^{-\gamma}}{p_0 \rho_0^{-\gamma}}$$

a quantity which turns out to vanish in first and even second order.

Along the segment S_1 the entropy is not exactly constant since the different particles on S_1 have crossed the first shock front at different places and thus acquired a different change in entropy. Since, however, the entropy change is negligibly small anyway, one may well assume the entropy constant on S_1 . No great error is indeed committed by determining the density ρ in S_1 from

$$\rho p^{-\frac{1}{\gamma}} = \rho_1 p_1^{-\frac{1}{\gamma}} = \rho_0 p_0^{-\frac{1}{\gamma}}.$$

The enthalpy p. u. mass along S_1 can then be determined from

$$1 = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}.$$

The flow in region (1) is neither quite isentropic nor quite irrotational since the strength of the first shock front varies. If it were, Bernoulli's equation

$$1 + \frac{1}{2}(u + U)^2 + \frac{1}{2}v^2 = i_1 + \frac{1}{2}(u_1 + U)^2 + v_1^2 = i_0 + \frac{1}{2}U^2$$

would be valid. Since, however, the entropy change is of third order, no great error is committed in assuming Bernoulli's law to be valid along S_1 . Relation III' then is simplified to

$$\text{III}'' \quad 2\pi i_0 \int_{S_1+S_2} \rho v r dx - 2\pi U \int_{S_1+S_2} \rho u v r dx - 2\pi i_0 \rho_0 U^2 \epsilon_3 = F U$$

It would seem that knowledge of the radial component ^{and of u} v along $S_1 + S_2$ were necessary to verify III''. The two integrals, however, can be directly determined from relations I and II. Relation III'' then reduces to

$$\rho_0 = \rho_2$$

and relation I assumes the simpler form

$$\text{I}' \quad 2\pi \int_{S_1+S_2} \rho v r dx = \rho_0 U \epsilon_3.$$

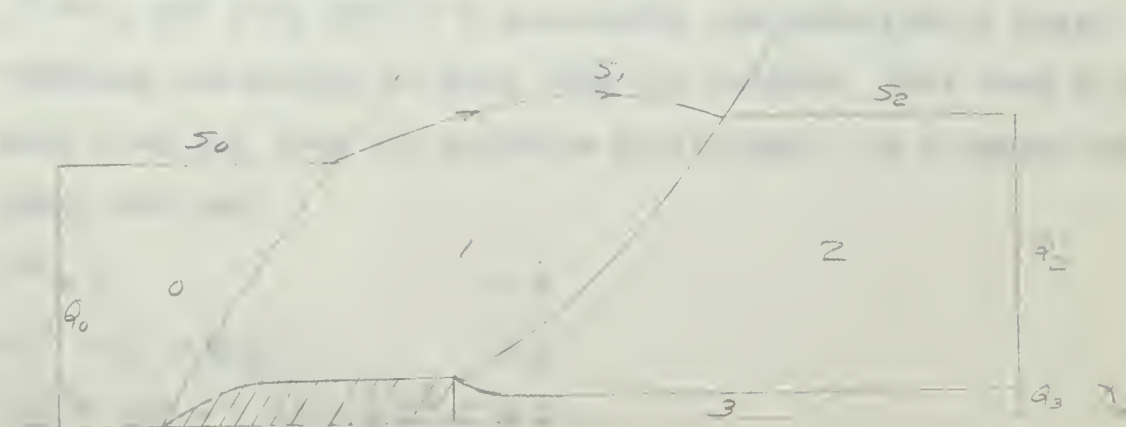
Hence, under the various simplifying assumptions made relations I' and II are the only ones to be verified. They involve the velocities (u, v) along S_1 but not the pressures. Knowledge of the pressure would, however, only furnish approximate knowledge of $(U + u)^2 + v^2$ by Bernoulli's law.

It thus appears that a verification of the conservation laws in which the experimental pressure distribution enters is not possible without knowledge of the velocity distribution.

I thought it reasonable to use for S_1 a line parallel to the axis since I understand that the pressure distribution is measured on such a line (or what is equivalent, at a fixed point at various times).

Instead, one might try to use for S_1 a line the tangent of which at each point has just the flow direction. This is, however, not possible owing to the peculiar behavior of the velocity vectors between the two shock fronts. The essential feature of this behavior, indicated in Figure 1, is that the velocity vector turns through about 180° along S_1 . That this is so follows from the fact that, on passing the first shock front, the air is set in motion while on passing the second front the air is nearly stopped.

Another possibility might be considered; viz., to use for $S_1 + S_2$ a line which would be the streamline when the flow is considered from the moving bullet. Such a line runs about as indicated. The laws of conservation assume a particularly simple form when referred to this line.



The reason is that the flux p. u. time of mass, momentum, or energy across this line is balanced by the change p. u. time of these quantities resulting from the fact that the section of the region (1) cut out by $S_0 + S_1 + S_2$ changes in time.

The formulas I, II, III' become

$$I \quad -\rho_0 U Q_0 + \rho_2 U Q_2 = 0$$

$$II \quad p_0(Q_0 - Q_2 - Q_3) + 2\pi \int_{S_1+S_2} p r \frac{dr}{dx} dx = F$$

$$III' \quad -\frac{e}{c} \rho_0 U(Q_0 - Q_2) - p_0 U Q_3 = FU$$

The formulas are very simple but to verify them it would again be necessary to know the directions of the flow in order to be able to draw the particular lines S_1 and S_2 assumed.

That no terms referring to the flux across $S_1 + S_2$ occur follows from the fact that, as seen from the bullet, no such flux occurs and that the three conservation laws

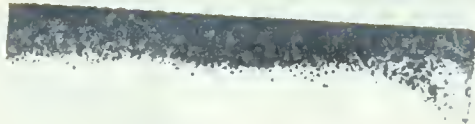
$I = 0$, $II = F$, $III = FU$ are equivalent to the laws

$I^* = 0$, $II^* = F$, $III^* = 0$ expressing conservation of mass, momentum and energy as seen from the bullet. That does not mean that the laws are pairwise equivalent. As a matter of fact, one has

$$I^* = I = 0$$

$$II^* = II - U \cdot I = F$$

$$III^* = III - U \cdot II + \frac{1}{2} U^2 \cdot I = 0$$



the fact

Clearly, III^{*} expresses that the bullet does no work when the observer moves with it. The energy is drawn from the incoming flow.

I have always found the three laws I^{*}, II^{*}, III^{*} much easier to derive and I have become accustomed to thinking in terms of them since they are equivalent to I, II, III anyway.

I am sorry that I do not see a way to derive formulas which could be verified by using the experimentally found pressure distribution. Nevertheless, I hope that my remarks will have some interest for you.

With best regards, I am

Very sincerely yours,

K. O. Friedrichs

P. S. I just read with great pleasure your report on rocket efficiencies plus appendices.

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